Solution to Problem #5

$$\mu\frac{\partial^2\xi}{\partial t^2} + YI\frac{\partial^4\xi}{\partial z^4} = 0\,,$$

subject to boundary conditions

$$\xi\Big|_{z=0} = \frac{\partial\xi}{\partial z}\Big|_{z=0} = 0\,,$$

$$\frac{\partial^2\xi}{\partial z^2}\Big|_{z=L} = \frac{\partial^3\xi}{\partial z^3}\Big|_{z=L} = 0\,,$$

at the ends.

Having deduced the equation of motion and the boundary conditions, we now proceed to set

$$\xi(z,t) = \zeta(z)\, e^{i\omega t},$$

giving

$$\left(\frac{d^4}{dz^4} - k^4\right)\zeta(z) = 0\,, \text{ where } k^4 = \frac{\mu\omega^2}{YI}\,.$$

The fourth–order differential equation can be rewritten

$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{d^2}{dz^2} + k^2\right)\zeta(z) = 0,$$

leading to 4 independent solutions

$$\zeta(z) = \begin{cases} \sin kz \\ \cos kz \\ \sinh kz \\ \cosh kz \end{cases}.$$

We thus express the displacement of the vibrating rod as

$$\zeta(z) = A\sin kz + B\cos kz + C\sinh kz + D\cosh kz\,.$$

Setting $\zeta(0) = \zeta'(0) = 0$ we have

$$k\left(A + C\right) = B + D = 0,$$

or

$$\zeta(z) = A(\sin kz - \sinh kz) + B(\cos kz - \cosh kz)\,.$$

Applying the conditions $\zeta''(L) = \zeta'''(L) = 0$ we find

$$-k^2 A(\sin kL + \sinh kL) - k^2 B(\cos kL + \cosh kL) = 0\,,$$

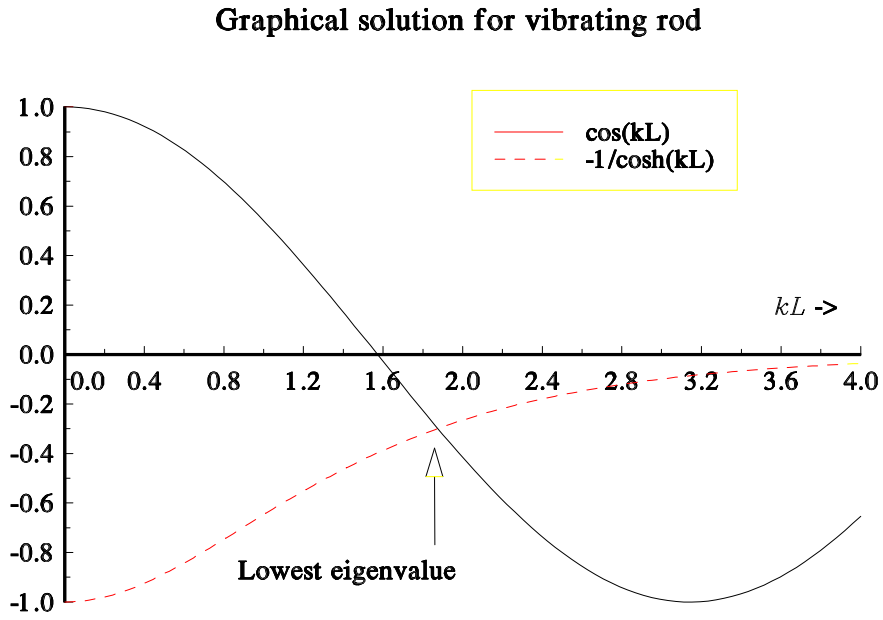
$$-k^3 A(\cos kL + \cosh kL) + k^3 B(\sin kL - \sinh kL) = 0\,.$$

The eigenvalue condition (secular equation) is then

$$\det \begin{vmatrix} \sin kL + \sinh kL & \cos kL + \cosh kL \\ -(\cos kL + \cosh kL) & \sin kL - \sinh kL \end{vmatrix} = 0 ,$$

or, with $\theta = kL$,

$\cos\theta \cosh\theta = -1$.



Using the hybrid root-finder routine `ansfalsi.f` I got the following results:

```
: f1   fdup  fcos  fswap  fcosh  f*  1e0 f+  ;  ok
use( f1 0e0 3e0 1e-8 )falsi fs. 1.87510E0  ok
use( f1 3e0 5e0 1e-8 )falsi fs. 4.69409E0  ok
use( f1 5e0 7e0 1e-8 )falsi fs.
Error: )falsi EVEN # OF ROOTS IN INTERVAL!
use( f1 5e0 8e0 1e-8 )falsi fs. 7.85476E0  ok
```

That is, these are the first 3 roots of $\cos\theta \cosh\theta + 1 = 0$. Young's modulus for Al is

$Y = (5.6\text{–}7.7) \times 10^{11}$ dyne/cm^3 ,

the lineal density is $\mu = \rho \pi r^2$, and $I = \pi r^4/4$. Thus the frequencies in Hz are

$\nu_1 = 9.5\text{–}11.3$

$\nu_2 = 59.5\text{–}70.5$

$\nu_3 = 167\text{–}197$